

On One Problem of Optimization of Approximate Integration *

V. F. Babenko

Abstract

It is proved that interval quadrature formula of the form

$$q(f) = \sum_{k=1}^n c_k \frac{1}{2h} \int_{x_k-h}^{x_k+h} f(t) dt$$

($c_k \in \mathbb{R}$, $x_1 + h < x_2 - h < x_2 + h < \dots < x_n - h < x_n + h < x_1 + 2\pi - h$) with equal c_k and equidistant x_k is optimal among all such formulas for the class $K * F_1$ of convolutions of a *CVD*-kernel K with functions from the unite ball of the space L_1 of 2π -periodic integrable functions.

Key words: interval quadrature formula, *CVD*-kernel, classes of convolutions.

1. Let C and L_p ($1 \leq p \leq \infty$) be the spaces of 2π -periodic functions endowed with corresponding norms; $\|\cdot\|_p$ - norm in L_p . The convolution of functions $K \in L_1$ (kernel of convolution) and $\phi \in L_1$ is defined by equality

$$K * \phi(x) = \int_0^{2\pi} K(x-t)\phi(t)dt.$$

Given a kernel K set $\mu = \mu(K) = 1$, if $\int_0^{2\pi} K(t)dt = 0$, and $\mu = \mu(K) = 0$, if $\int_0^{2\pi} K(t)dt \neq 0$. Denote by $\nu(f)$ the number of sign changes over a period of a 2π -periodic function f .

A kernel K is called a *CVD*-kernel (denoted by $K \in CVD$), if for any function of the form

$$f(x) = a\mu + K * \phi(x) \tag{1}$$

($a \in \mathbb{R}$, $\phi \in C$, $\phi \perp \mu$) the inequality $\nu(f) \leq \nu(\phi)$ holds.

Let $\psi(x)$ be an entire function of the form

$$\psi(x) = x^l e^{-\gamma x^2 + \delta x} \prod_{k=1}^{\infty} (1 + \delta_k x) e^{\delta_k x}$$

*This paper was published in Russian in Studies in Modern Problems of Summation and Approximation of Functions and Their Applications, Collect. Sci. Works, Dnepropetrovsk, 1984, pp. 3 -13

where $l \in \mathbb{Z}_+$, $\gamma, \delta, \delta_k \in \mathbb{R}$, $0 < \gamma^2 + \sum_{k=1}^{\infty} |\delta_k| < \infty$. Then (see [1], [2]) the kernel

$$K(x) = \sum_{k=-\infty}^{\infty}{}' \frac{e^{ikx}}{\psi(ik)}$$

($\sum_{k=-\infty}^{\infty}{}'$ denotes that the summation is carried out over all k such that $\psi(ik) \neq 0$) is a *CVD* – kernel. In particular, Bernuolly's kernels

$$B_r(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty}{}' \frac{e^{ikx}}{(ik)^r}$$

and, more generally, kernels of the form

$$B_{\mathcal{P}}(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty}{}' \frac{e^{ikx}}{\mathcal{P}(ik)},$$

where $\mathcal{P}(x)$ is an algebraic polynomial having real zeros (integral operator of convolution with such a kernel inverses differential operators of the form $\mathcal{P}(\frac{d}{dx})$) are *CVD*-kernels.

Denote by F_p ($1 \leq p \leq \infty$) the unite ball in the space L_p . Given a kernel K denote by $K * F_p$ the class of functions f of the form (1) where $a \in \mathbb{R}$, $\phi \in F_p$, $\phi \perp \mu$. Note that $B_r * F_p$ is standard for the theory of quadrature formulas class W_p^r of real-valued, 2π -periodic functions f having locally absolutely continuous derivative $f^{(r-1)}$ ($f^{(0)} = f$) and such that $\|f^{(r)}\|_p \leq 1$.

Consider the set Q_n ($n = 1, 2, \dots$) of all possible quadrature formulas of the form

$$q(f) = \sum_{k=1}^n c_k f(x_k), \quad (2)$$

where $c_k \in \mathbb{R}$, $x_1 < x_2 < \dots < x_n < x_1 + 2\pi$. The problem about optimal for the class $K * F_p$ quadrature formula from Q_n is formulated in the following way. Find the value

$$R_n(K * F_p) = \inf_{q \in Q_n} \sup_{f \in K * F_p} \left| \int_0^{2\pi} f(x) dx - q(f) \right|, \quad (3)$$

and parameters (knots x_k and coefficients c_k) of a quadrature formula q that realizes inf in the right hand part of (3).

This problem was completely solved for the classes W_p^r ($r = 1, 2, \dots$; $1 \leq p \leq \infty$) in the papers of V. P. Motornyi [3], A. A. Ligun [4], A. A. Zhensykbayev [5], [6]. It was proved that for any n the optimal formula has n equidistant knots and equal coefficients. In the papers [7], [8], [9], [10] these results were generalized to the cases of more general function classes $K * F_p$.

We will consider the following problem. Let *CVD*-kernel K ; $p \in [1, \infty]$; $n = 1, 2, \dots$; $h \in (0, \pi/n)$ be given. Denote by $Q_{n,h}$ the set of all functionals of the form

$$q(f) = \sum_{k=1}^n c_k \frac{1}{2h} \int_{x_k-h}^{x_k+h} f(t) dt \quad (4)$$

where $c_k \in \mathbb{R}$, $x_1 + h < x_2 - h < x_2 + h < \dots < x_n - h < x_n + h < x_1 + 2\pi - h$. Set

$$\begin{aligned} R(f, q) &= \int_0^{2\pi} f(t) dt - q(f), \\ R(K * F_p, q) &= \sup_{f \in K * F_p} |R(f, q)|, \\ R_{n,h}(K * F_p) &= \inf_{q \in Q_{n,h}} R(K * F_p, q). \end{aligned} \quad (5)$$

The problem is formulated as follows. Find the value (5) and parameters x_k and c_k ($k = 1, \dots, n$) of a functional q that realizes inf in the right hand part of (5).

From the applications point of view, interval quadrature formulae are more natural than the usual quadrature formulae based on values at points, since quite often the result of measuring physical quantities, due to the structure of the measurement devices, is an average values of the function, describing the studied quantities, over some interval. Note that one can obtain the usual quadrature formula from the corresponding interval quadrature formula as a limit case, setting $h \rightarrow 0$.

In this paper we solve the problem on optimal interval quadrature formula for the class $K * F_p$ with $p = 1$. Thus we will essentially use Ligun's idea from [4].

2. Suppose that $f \in K * F_1$, $q \in Q_{n,h}$, and obtain for $R(f, q)$ an integral representation. Given $q \in Q_{n,h}$ let

$$H(q; t) = \frac{1}{2h} \sum_{k=1}^n c_k \chi_{(x_k-h, x_k+h)}(t)$$

where χ_A is the indicator of a set $A \subset [0, 2\pi]$ continued with the period 2π on the entire real axis. We will have

$$\begin{aligned} R(f, q) &= a \cdot \mu \left(2\pi - \sum_{k=1}^n c_k \right) + \int_0^{2\pi} \phi(u) \left[\int_0^{2\pi} K(t-u) dt - \int_0^{2\pi} K(t-u) H(q; t) dt \right] du \\ &= a \cdot \mu \left(2\pi - \sum_{k=1}^n c_k \right) + \int_0^{2\pi} \phi(u) \int_0^{2\pi} K(t-u) [1 - H(q; t)] dt du. \end{aligned}$$

Let

$$M(q, u) = \int_0^{2\pi} K(t-u) [1 - H(q; t)] dt.$$

We obtain

$$R(f, q) = a \cdot \mu \left(2\pi - \sum_{k=1}^n c_k \right) + \int_0^{2\pi} \phi(u) M(q, u) du. \quad (6)$$

In the case $\mu(K) = 0$ the first term in the right-hand part of (6) is equal to zero. Solving the problem about optimal interval quadrature formula for the class $K * F_1$ in the case $\mu(K) = 1$ we can consider functionals q such that $\sum_{k=1}^n c_k = 2\pi$ only (otherwise

$R(K * F_1, q) = +\infty$). Thus in the case $\mu(K) = 1$ we can assume that the first term in the right-hand part of (6) is equal to zero also.

Taking into account the relations (6), above presented facts, and S. M. Nikol'skii's duality theorem (see [11], [12, Chapt. 2, Theorem 2.2.1]) we obtain

$$R(K * F_1; q) = \begin{cases} \inf_{\lambda \in \mathbb{R}} \|M(q; \cdot) - \lambda\|_\infty, & \text{if } \mu(K) = 1, \\ \|M(q; \cdot)\|_\infty, & \text{if } \mu(K) = 0. \end{cases} \quad (7)$$

For $n = 1, 2, \dots$; $\lambda \in \mathbb{R}$, and $0 < h < \pi/n$ set

$$q_{n,h,\lambda}(f) = \lambda \sum_{k=1}^n \frac{1}{2h} \int_{2k\pi n^{-1}-h}^{2k\pi n^{-1}+h} f(t) dt.$$

Theorem 1. *Let K be a CVD-kernel, $n = 1, 2, \dots$, and $0 < h < \pi/n$. Then*

$$R_{n,h}(K * F_1) = \begin{cases} R(K * F_1; q_{n,h,2\pi/n}) = \inf_{\lambda \in \mathbb{R}} \|M(q_{n,h,2\pi/n}; \cdot) - \lambda\|_\infty, & \text{if } \mu(K) = 1, \\ R(K * F_1; q_{n,h,\lambda}) = \inf_{\lambda \in \mathbb{R}} \|M(q_{n,h,\lambda}; \cdot)\|_\infty, & \text{if } \mu(K) = 0. \end{cases}$$

Proof. Case $\mu(K) = 1$. Denote by λ_q ($q \in Q_{n,h}$) the constant of the best L_∞ -approximation of the function $M(q; \cdot)$. Suppose that for some $q \in Q_{n,h}$ the inequality

$$\|M(q; \cdot) - \lambda_q\|_\infty < \|M(q_{n,h,2\pi/n}; \cdot) - \lambda_{q_{n,h,2\pi/n}}\|_\infty$$

holds true. Let

$$\Delta_\tau(t) = M(q; t - \tau) - M(q_{n,h,2\pi/n}; t) - \lambda_q + \lambda_{q_{n,h,2\pi/n}}, \quad \tau \in \mathbb{R}.$$

Since the function $M(q_{n,h,2\pi/n}; \cdot)$ is $2\pi/n$ -periodic, we will have that for any τ

$$\nu(\Delta_\tau(\cdot)) \geq 2n.$$

It is clear that $K(-\cdot) \in CVD$ with $\mu = 1$. Taking into account this fact and representation (6) it is easy to verify that for any τ we will have

$$\nu(\Delta'_\tau(\cdot)) \geq 2n$$

where

$$\Delta'_\tau(t) = -H(q_{n,h,2\pi/n}; t) - H(q; t - \tau).$$

Among the coefficients c_k of the functional q choose such that $|c_k| \leq 2\pi/n$ and denote by k_0 its number. Set $\tau_0 = 2\pi/n - x_{k_0}$.

Lemma 1.

$$\nu(\Delta_{\tau_0}(\cdot)) \leq 2n - 2.$$

If Lemma 1 will be proved, we obtain the contradiction with the above presented statement about the number of the sign changes of Δ'_τ . After that the part of Theorem 1 related to the case $\mu(K) = 1$ will be proved.

Proof of the Lemma 1. Suppose that $\nu(\Delta'_{\tau_0}) \geq 2n$. It means that there exist points $y_1 < y_2 < \dots < y_{2n} < y_1 + 2\pi$ such that difference Δ'_{τ_0} has at these points nonzero values of alternating sign. Let (for definiteness) Δ'_{τ_0} has negative values at the points y_2, y_4, \dots, y_{2n} . It is easily seen that such points must belong to the different intervals of positivity of the function $H(q; t - \tau_0)$. But the number of such intervals inside an interval of the length 2π is less than or equal to n . Really, the difference Δ'_{τ_0} can not change sign inside the common part of an interval of positivity of the function $H(q_{n,h,2\pi/n}; t)$ and an interval of positivity of the function $H(q; t - \tau_0)$ (this difference is constant inside such an part). Thus at least one of the point of negativity (say y_{2l}) must belongs to the interval $(2\pi/n - h, 2\pi/n + h)$. But in view of the choice of τ_0 , the difference Δ'_{τ_0} is nonnegative on this interval. This is a contradiction.

Lemma is proved.

The case $\mu(K) = 0$. We need the following analog of the Lemma 1 from [8].

Lemma 2. Let λ^* be such that

$$\inf_{\lambda \in \mathbb{R}} \|M(q_{n,h,\lambda}; \cdot)\|_\infty = \|M(q_{n,h,\lambda^*}; \cdot)\|_\infty.$$

Then

$$\max_t M(q_{n,h,\lambda^*}; t) = -\min_t M(q_{n,h,\lambda^*}; t).$$

Proof of Lemma 2. We have

$$M(q_{n,h,\lambda^*}; t) = \int_0^{2\pi} K(u) du - \lambda \int_0^{2\pi} K(u-t) H(q_{n,h,1}; u) du.$$

It is well known that CVD -kernel K with $\mu(K) = 0$ does not change sign (let it is nonnegative for definiteness). Then it is easily seen that function

$$\psi(t) = \int_0^{2\pi} K(u-t) H(q_{n,h,1}; u) du$$

is strictly positive. If for a given $\lambda_0 > 0$

$$\max_t \left[\int_0^{2\pi} K(u) du - \lambda_0 \psi(t) \right] > -\min_t \left[\int_0^{2\pi} K(u) du - \lambda_0 \psi(t) \right], \quad (8)$$

then for $\lambda < \lambda_0$ and close enough to λ_0 we will have

$$\left\| \int_0^{2\pi} K(u) du - \lambda \psi(\cdot) \right\|_\infty < \left\| \int_0^{2\pi} K(u) du - \lambda_0 \psi(\cdot) \right\|_\infty. \quad (9)$$

If instead of (8) we have

$$\max_t \left[\int_0^{2\pi} K(u) du - \lambda_0 \psi(t) \right] < - \min_t \left[\int_0^{2\pi} K(u) du - \lambda_0 \psi(t) \right],$$

then the inequality (9) will hold true for all $\lambda > \lambda_0$ and close enough to λ_0 . Since the existence of λ^* and its positiveness are obvious, the Lemma is proved.

Suppose that for some $q \in Q_{n,h}$

$$R(K * F_1; q) < \inf_{\lambda \in \mathbb{R}} R(K * F_1; q_{n,h,\lambda}) \quad (10)$$

or that is equivalent (in view of (7))

$$\|M(q; \cdot)\|_\infty < \|M(q_{n,h,\lambda^*}; \cdot)\|_\infty. \quad (11)$$

Taking into account the Lemma 2 and the fact that $M(q_{n,h,\lambda^*}; \cdot)$ is $2\pi/n$ -periodic, we conclude that

$$\nu(\Delta_\tau(\cdot)) \geq 2n.$$

where

$$\Delta_\tau(t) = M(q; t - \tau) - M(q_{n,h,\lambda^*}; t)$$

But

$$\Delta_\tau(t) = \int_0^{2\pi} K(u - t) [H(q_{n,h,\lambda^*}; u) - H(q; u - \tau)] du.$$

Since $K \in CVD$ for any τ then

$$\nu(\Delta'_\tau) := \nu(H(q_{n,h,\lambda^*}; u) - H(q; u - \tau)) \geq 2n.$$

However analogously to the case $\mu(K) = 1$ it is possible to choose τ_0 such that $\nu(\Delta'_{\tau_0}) \leq 2n - 2$. We omit the details.

Therefore the relation (11), and consequently the relation (10), is impossible. Theorem 1 is proved.

Let us show that Theorem 1 implies the optimality of the quadrature formula with equidistant knots and equal coefficients on the class $K * F_1$ among the all quadrature formulas from Q_n . Let a CVD -kernel K be such that the class $K * F_1$ is a relatively compact subset of the space C (in the case $\mu(K) = 0$) or can be obtained by shifts to the constants from relatively compact subset of C (in the case $\mu(K) = 1$). For any quadrature formula q of the form (2) let $\{q_h, h > 0\}$ be the family of functionals of the form (4) having the same c_k and x_k . It is easily seen that $q_h(f) \rightarrow q(f)$ uniformly on the set $f \in K * F_1$ as $h \rightarrow 0$ (remind that in the case $\mu(K) = 1$ we consider formulas q and functionals q_h such that $\sum_{k=1}^n c_k = 2\pi$ only). Set

$$q_{n,0,\lambda}(f) = \lambda \sum_{k=1}^n f\left(\frac{2k\pi}{n}\right).$$

Then (we restrict ourself by the case $\mu(K) = 1$) for any $q \in Q_n$ we will have

$$R(K * F_1; q) = \lim_{h \rightarrow 0} R(K * F_1; q_h) \geq \lim_{h \rightarrow 0} R(K * F_1; q_{n,h,2\pi/n}) = R(K * F_1; q_{n,0,2\pi/n})$$

that is the optimality of the rectangle formula on the class $K * F_1$.

Finely we note that the statement of the Theorem 1 can be easily generalized to the classes of convolutions of the functions from F_1 with $O(M, \Delta)$ -kernels (see [8]).

References

- [1] J. C. Mairhuber, I. J. Schoenberg, R. E. Williamson, On variation diminishing transformations on the circle, *Rend. Circolo math. Palermo*, 1959, 8, N 3, p. 241 - 270.
- [2] K. I. Babenko, Some questions of approximate definition and calculation of functions, *Moskow, IPM AN SSSR*, 1970
- [3] V. P. Motornyi, On the best quadrature formulas of the form $\sum_{k=1}^n p_k f(x_k)$ for some classes of differentiable periodic functions, *Izv. AN SSSR, ser. Mat.*, 1974, v. 38, 3, p. 583 - 614.
- [4] A. A. Ligun, Sharp inequalities for spline-functions and optimal quadrature formulas for certain function classes, *Mat. Zametki*, 1976, V. 19, 6, p. 913-926.
- [5] A. A. Zhensykbayev, The best quadrature formula for some classes of differentiable periodic functions, *Izv. AN SSSR, ser. Mat.*, 1977, v. 41, 5, p. 1110 - 1124.
- [6] A. A. Zhensykbayev, Monosplines of minimal norm and the best quadrature formulas, *Uspehi Mat. Nauk*, 1981, v. 36, 4, p. 107-159.
- [7] T. A. Grankina, On the best quadrature formulas for some convolution classes, 15 p., *Dep v VINITI* 22.12.1981, N 5782-81.
- [8] V. F. Babenko, T. A. Grankina, On the best quadrature formulas for classes of convolutions with $O(M, \Delta)$ -kernels, *Studies on modern problems of summation and approximation of functions and their applications*, *Collect. Sci. Works, Dnepropetrovsk*, 1982, p. 6-13.
- [9] M. A. Chahkiev, On optimality of equidistant knots, *Dokl. AN SSSR*, 1982, 264, 4, p. 836-839.
- [10] M. A. Chahkiev, Linear differential operators and optimal quadrature formulas, *Dokl. AN SSSR*, 1983, v. 273, 1, p. 60 - 65.
- [11] S. M. Nikol'skii, Approximation of functions by trigonometric polynomials in the mean, *Izv. AN SSSR, Ser. Mat.*, 1946, v. 3, p. 207 - 256.
- [12] N. P. Korneichuk, *Extremal problems of Approximation Theory*, *Moskow, Nauka*, 1976.